International Conference on Control, Engineering & Information Technology (CEIT'13)
Proceedings Engineering & Technology - Vol.3, pp. 189-193, 2013

Exponential Synchronization for a class of Nonlinear Dynamical Networks

Houda KHELIFI

ISSAT, Sousse University

houda.khlifi@gmail.com

***Abstract*— In this paper, the exponential synchronization of a class of chaotic neural networks with delays using the drive-reponse concept is investigated .**

Based on the Lyapunov stability method and the Halanay inequality lemma, a delay independent sufficient exponential synchronization condition is derived. The synchronization condition can be easily verified and implemented.

Finally, some illustrative examples are given to demonstrate the effectiveness of the presented synchronization scheme.

***Keywords*— synchronization, chaos, chaotic neural networks, exponential synchronization.**

I. INTRODUCTION

A chaotic system is a nonlinear dynamical system, which has several properties such as the sensitivity to initial conditions as well as an irregular unpredictable behavior.

Chaos phenomenon has been applied in many disciplines such as secure communications, biological science neural networks, automatic control, etc [1].

Over the last decade, many new types of synchronization have appeared for example: chaotic synchronization [2, 3], lag synchronization [4], adaptive synchronization [5], phase synchronization [6], and generalized synchronization [4].

In 1990, Pecora and Carroll [2], proposed drive-reponse concept for constructing synchronization of coupled chaotic systems. The synchronization of coupled chaotic systems has received considerable attention [5, 7, 8]. Especially, a typical study of synchronization is the coupled identical chaotic systems [6].

Recently, there is increasing interest in the study of dynamical properties of delayed neural networks (DNNs) [9-13, 21-23]. Most previous studies have concentrated on the stability analysis and periodic oscillations of this kind of networks. However, it has been shown that such networks can exhibit some complicated dynamics and even chaotic behaviors. In particular, the introduction of delays into neural networks makes their dynamical behaviors much more complicated [13-18]. Furthermore, there are few studies in the synchronization issue for this class of chaotic neural networks with delay [19].

This work, addresses the synchronization problem of a class of chaotic neural networks with delays. Based on the Lyapunov stability method and the Halanay inequality lemma, a delay independent sufficient exponential synchronization condition is derived. This paper is organized as follows. In Section II, we provide a description of chaotic neural networks considered in the work and defines the exponential synchronization problem of the drive-response chaotic neural networks. In Section III we derive a control law to solve the synchronization problem, and a sufficient criterion for the exponential synchronization is established. In Section IV, we present some illustrative examples. Finally, we draw conclusions in Section V.

The following notations are used throughout this paper.

For $x \in \mathbb{R}^n$, let $\|x\| = \left(x^T x\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} x_i^2\right)^{\frac{1}{2}}$ denote the Euclidian vector norm.

Besides, and for a matrix $A \in \mathbb{R}^{n\times n}$, let $\|A\|$ indicate the norm of A induced by the Euclidean vector norm, i.e.,

$$\|A\| = \left(\lambda_{\max}(A^T A)\right)^{\frac{1}{2}},$$

Where $\lambda_{\max}(A)$ represents the maximum eigenvalue of matrix A and T denotes the transpose of a matrix. Note that, For all $(n\times n)$ real symmetric matrix A, one has A is positive definite if and only if all its eigenvalues are positive. Furthermore, for all $x \in \mathbb{R}^n$

$$\lambda_{\min}(A)\|x\|^2 \le x^T A x \le \lambda_{\max}(A)\|x\|^2$$

Where $\lambda_{\min}(A)(\lambda_{\max}(A))$ represents the minimum (resp. the maximum) eigenvalue of matrix A.

II. SYSTEMS DESCRIPTION AND SYNCHRONIZATION PROBLEM

A class of the delayed chaotic neural network considered in this paper is described by the following state equations:

$$\dot{x}(t) = -Dx(t) + Ag(x(t)) + \sum_{k=1}^{r} W_k g(x(t-\tau_k(t))) + J \tag{1}$$

Where $x(t)$ is the neural state vector, the matrix $D=diag\{d_1,d_2,...,d_n\}$ and $d_i>0$, $A\in\mathbb{R}^{n\times n}$, $W_k=\left(w_{ij}^k\right)_{n\times n}$, $k=1,2,...,r$ are the connection weight matrices, $g(x(t))\in\mathbb{R}^n$ denotes the neuron activation function with $g(0)=0$, J is a constant input to set the desired equilibrium point, and $\tau_k(t)$ is the constant discrete time delay.

Throughout this paper, we make the following assumptions.

(A1) Each function $g_i:\mathbb{R}\to\mathbb{R},i\in\{1,2,...,n\}$ is bounded, and satisfies the Lipschitz condition with a Lipschitz constant $L_i>0,i.e.\left|g_i(u)-g_i(v)\right|\le L_i\left|u-v\right|$ for all $u,v\in\mathbb{R}$.

(A2) $\tau_i(t)\ge 0$ is the delay function for all $1\le i\le n$. We will consider the euclidean norm in the whole the paper.

Let the chaotic system (1) be the drive system and it's unidirectionally coupled copy:

$$\dot{z}(t)=-Dz(t)+Ag(z(t)) + \sum_{k=1}^{r} W_k g(z(t-\tau_k(t)))+J+u(t) \qquad (2)$$

be the reponse system, where $u(t)$ denotes external control input that will be appropriately designed for a certain control objective.

Let $e_i(t)=x_i(t)-z_i(t)$ be the error between the two systems.

$e(t)\to 0$, as $t\to 0$ means that the drive neural networks and the response neural networks are synchronized.
Therefore, the error dynamics between (1) and (2) can be expressed by:

$$\dot{e}(t)=-D(x(t)-z(t))+A(g(x(t))-g(z(t)))+ \sum_{k=1}^{r} W_k(g(x(t-\tau_k(t)))-g(z(t-\tau_k(t))))-u(t) \qquad (3)$$

Or by the following compact form:

$$\dot{e}(t)=-De(t)+Ag(e(t))+Wg(e(t-\tau(t)))-u(t) \qquad (4)$$

.

If the state variables of the drive system are used to drive the response system, then the control input vector with state feedback is designed as follows:

$$\begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} w_{1j}(x_j(t)-z_j(t)) \\ \vdots \\ \sum_{j=1}^{n} w_{nj}(x_j(t)-z_j(t)) \end{bmatrix} = \begin{pmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{pmatrix} \begin{bmatrix} x_1(t)-z_1(t) \\ \vdots \\ x_n(t)-z_n(t) \end{bmatrix} = \Omega \begin{bmatrix} e_1(t) \\ \vdots \\ e_n(t) \end{bmatrix} \qquad (5)$$

Where Ω is the controller gain matrix and will be appropriately chosen for exponentially synchronizing both drive system and response system. With the control law (5), the error dynamics can be expressed by the following compact form:

$$\dot{e}(t)=-De(t)+Ag(e(t))+Wg(e(t-\tau(t)))-\Omega e(t) \qquad (6)$$

Definition 1. Systems (1) and (2) are said to be exponentially synchronized if there exist constants $\eta\ge 1$ and $\alpha>0$ such that for all $t\ge 0$

$$\left|x_i(t)-z_i(t)\right|\le\eta\left|x_i(0)-z_i(0)\right|\exp(-\alpha t)$$

Moreover, the constant α is defined as the exponential synchronization rate.

III. SYNCHRONIZATION CRITERION

In this section, using the Halanay inequality lemma, we establish a sufficient condition for synchronization of chaotic systems with delays.

Lemma 1 (Halanay inequality lemma [20]). Let $\tau\ge 0$ be a constant, and $V(t)$ be a non-negative continuous function defined for $[-\tau,+\infty[$ which satisfies for $t\ge 0$ $\dot{V}(t)\le -pV(t)+q(\sup_{t-\tau\le s\le 0}V(s))$, where p and q are constants. If p > q > 0, then for t > 0

$V(t)\le(\sup_{-\tau\le s\le 0}V(s))\exp(-\delta t)$, where δ is a unique positive root of the equation $\delta=p-q\exp(\delta\tau)$.

Theorem. *For these drive-response chaotic neural networks (1) and (2) which satisfy assumptions (A1)-(A2). If the controller gain matrix* Ω *in (5) is real symmetric and positive definite, and satisfies* $L(\|A\|+\|W\|)/\min_{1\le i\le 0}(d_i)+\lambda_{\min}(\Omega)<1$,

Where $L=\max_{1\le i\le n}(L_i)$ *, then the exponential synchronization of systems (1) and (2) is obtained.*

Proof. Consider the following continuous function:

$$V(t)=\frac{1}{2}e(t)^T e(t)=\frac{1}{2}\|e(t)\|^2 \quad (7)$$

It can easily be verified that V(t) is a non-negative function over $[-\tau,+\infty[$ and that it is radially unbounded, i.e.

$V(t)\to+\infty$ as $\|e\|\to+\infty$.

Using the definition of $g(e(t-\tau))$ and assumption (A1) yields

$$\|g(e(t-\tau))\|^2=\sum_{i=1}^{n}g_i^2(e_i(t-\tau))$$
$$\le\sum_{i=1}^{n}L_i^2e_i^2(t-\tau)$$
$$\le L^2\|e(t-\tau)\|^2$$

And $\|g(e(t))\|^2\le L^2\|e(t)\|^2$

Let us evaluate the time derivative of V along the trajectory of (6) gives:

$$\dot V(t)=-e^TDe(t)+e^TAg(e(t))+e^TWg(e(t-\tau(t)))-e^T\Omega e(t)$$

$$\begin{aligned}&\le-\|e\|\|D\|\|e(t)\|+\|e\|\|A\|\|g(e(t))\|+\|e\|\|W\|\|g(e(t-\tau(t)))\|-\lambda_{\min}(\Omega)\|e\|^2\\
&\le-\min(d_i)\|e\|^2+L\|A\|\|e\|^2+L\|W\|\|e\|\|e(t-\tau(t))\|-\lambda_{\min}(\Omega)\|e\|^2\\
&\le-\min(d_i)\|e\|^2+L\|A\|\|e\|^2+L\|W\|(\tfrac{1}{2}\|e\|^2+\|e(t-\tau(t))\|^2)-\lambda_{\min}(\Omega)\|e\|^2\\
&\le-(2\min(d_i)-2L\|A\|-L\|W\|+2\lambda_{\min}(\Omega))\tfrac{1}{2}\|e\|^2+L\|W\|\|e(t-\tau(t))\|^2\\
&\le-(2\min(d_i)-2L\|A\|-L\|W\|+2\lambda_{\min}(\Omega))V(t)\\
&\quad+L\|W\|\max_{t-\tau\le s\le t}(V(s))\end{aligned} \quad (8)$$

Applying Lemma 1 to (8), it can be shown that if

$$L(\|A\|+\|W\|)/\min(d_i)+\lambda_{\min}(\Omega)<1$$

Then

$$V(t)\le(\sup_{-\tau\le s\le 0}V(s))\exp(-\delta t) \quad (9)$$

Where

$$\delta=(2\min(d_i)-2L\|A\|-L\|W\|+2\lambda_{\min}(\Omega))-L\|W\|\exp(\delta\tau) \quad (10)$$

Therefore, V(e(t)) converges to zero exponentially, which in turn implies that e(t) also converges globally and exponentially to zero with a convergence rate of $\delta/2$. Therefore, every trajectory $z_i(t)$ of (2) must synchronize exponentially toward the $x_i(t)$ with a convergence rate of $\delta/2$.

IV. APPLICATION

Example 1. Consider a delayed neural network as below:

$$\begin{bmatrix}\dot x_1\\ \dot x_2\end{bmatrix}=-\begin{bmatrix}1&0\\0&1\end{bmatrix}\times\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+\begin{bmatrix}2&-0.1\\-5&4.5\end{bmatrix}\times\begin{bmatrix}g_1(x_1(t))\\g_2(x_2(t))\end{bmatrix}+\begin{bmatrix}-1.5&-0.5\\-0.2&-4\end{bmatrix}\times\begin{bmatrix}g_1(x_1(t-1))\\g_2(x_2(t-1))\end{bmatrix} \quad (11)$$

Where $d_i=[1\ \ 1]^T$, $g_i(x_i)=\tanh(x_i)$,

$A=\begin{bmatrix}2&-0.1\\-5&4.5\end{bmatrix}$ and $W=\begin{bmatrix}-1.5&-0.5\\-0.2&-4\end{bmatrix}$

The system satisfies assumptions (A1) with $L_1=L_2=1$.

$\|A\|=6.9099$ and $\|W\|=4.0522$. Fig.1 shows the chaotic behavior of the system (11) with the initial condition $[x_1(s)\ \ x_2(s)]=[0.4\ \ 0.6]$ for $-1\le s\le 0$.

The response chaotic neural network with delays is designed by

$$\begin{bmatrix}\dot z_1\\ \dot z_2\end{bmatrix}=-\begin{bmatrix}1&0\\0&1\end{bmatrix}\times\begin{bmatrix}z_1(t)\\z_2(t)\end{bmatrix}+\begin{bmatrix}2&-0.1\\-5&4.5\end{bmatrix}\times\begin{bmatrix}g_1(z_1(t))\\g_2(z_2(t))\end{bmatrix}+\begin{bmatrix}-1.5&-0.5\\-0.2&-4\end{bmatrix}\times\begin{bmatrix}g_1(z_1(t-1))\\g_2(z_2(t-1))\end{bmatrix}+u(t) \quad (12)$$

If the controller gain matrix in (5) is chosen as

$\Omega=\begin{bmatrix}12&-4\\-4&20\end{bmatrix}$ with eigenvalues

$\lambda_{\min}(\Omega)=10.3431$ and $\lambda_{\max}(\Omega)=21.6569$, then the following inequality:

$$10.9621=L(\|A\|+\|W\|)<\min(d_i)+\lambda_{\min}(\Omega)=11.3431$$

is satisfied. It follows from the main theorem that the systems

(11) and (12) have been synchronized with an exponential convergence rate of 0.0792.

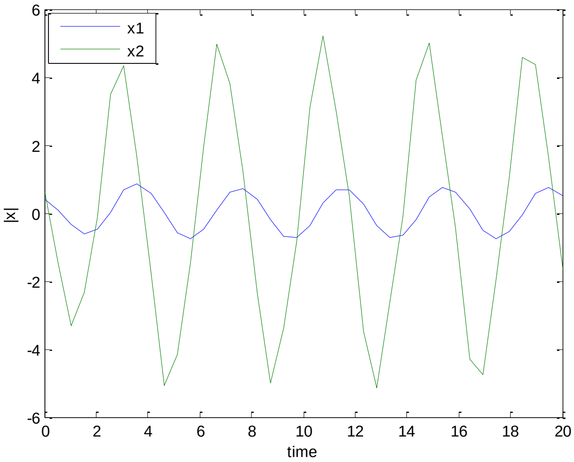

Fig.1. the chaotic behavior of the delayed NN in Example1.

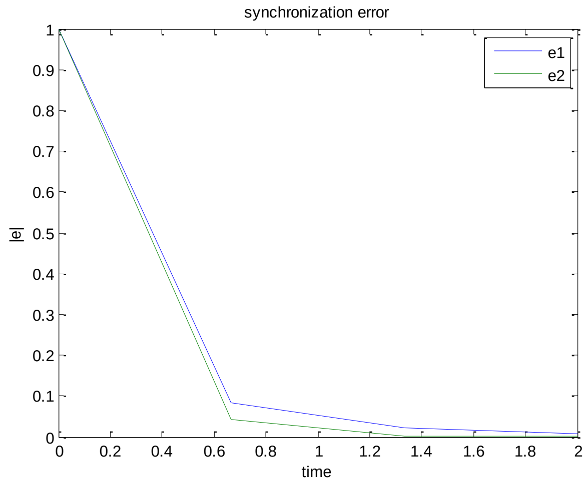

Fig. 2. The synchronization error (Example 1).

Example 2. Consider a chaotic neural network (NN) with delays as follows:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=-\begin{bmatrix}1&0\\0&1\end{bmatrix}\times\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+\begin{bmatrix}1+\frac{\Pi}{4}&20\\0.1&1+\frac{\Pi}{4}\end{bmatrix}\qquad(13)$$

$$\times\begin{bmatrix}g_1(x_1(t))\\g_2(x_2(t))\end{bmatrix}+\begin{bmatrix}-\sqrt{2}\frac{\Pi}{4}1.3&0.1\\0.1&-\sqrt{2}\frac{\Pi}{4}1.3\end{bmatrix}\times\begin{bmatrix}g_1(x_1(t-1))\\g_2(x_2(t-1))\end{bmatrix}$$

Where $d_i=[1\quad 1]^T$, $g_i(x_i)=(|x_i+1|-|x_i-1|)/2, i=1,2$,

$$A=\begin{bmatrix}1+\frac{\Pi}{4}&20\\0.1&1+\frac{\Pi}{4}\end{bmatrix}\text{ and }W=\begin{bmatrix}-\sqrt{2}\frac{\Pi}{4}1.3&0.1\\0.1&-\sqrt{2}\frac{\Pi}{4}1.3\end{bmatrix}$$

The system satisfies assumptions (A1) with $L_1=L_2=1$. $\|A\|=20.1589$ and $\|W\|=1.5439$ .Fig.3 shows the chaotic behavior of the system with the initial condition $[x_1(s)\quad x_2(s)]=[0.1\quad 0.1]$ for $-1\le s\le 0$.

the response chaotic neural network is designed as follows:

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}=-\begin{bmatrix}1&0\\0&1\end{bmatrix}\times\begin{bmatrix}z_1(t)\\z_2(t)\end{bmatrix}+\begin{bmatrix}1+\frac{\Pi}{4}&20\\0.1&1+\frac{\Pi}{4}\end{bmatrix}\times\begin{bmatrix}g_1(z_1(t))\\g_2(z_2(t))\end{bmatrix}\qquad(14)$$

$$+\begin{bmatrix}-\sqrt{2}\frac{\Pi}{4}1.3&0.1\\0.1&-\sqrt{2}\frac{\Pi}{4}1.3\end{bmatrix}\times\begin{bmatrix}g_1(z_1(t-1))\\g_2(z_2(t-1))\end{bmatrix}+u(t)$$

If the controller gain matrix in (5) is chosen as

$\Omega=\begin{bmatrix}24&-6\\-6&40\end{bmatrix}$ with eigenvalues $\lambda_{min}(\Omega)=22$

And $\lambda_{max}(\Omega)=42$ $\lambda_{max}(\Omega)=21.6569$, then the following inequality:

$21.7029=L(\|A\|+\|W\|)<\min(d_i)+\lambda_{min}(\Omega)=23$ is satisfied. It follows from the main theorem that the systems (13) and (14) have been synchronized with an exponential convergence rate of 0.3889.

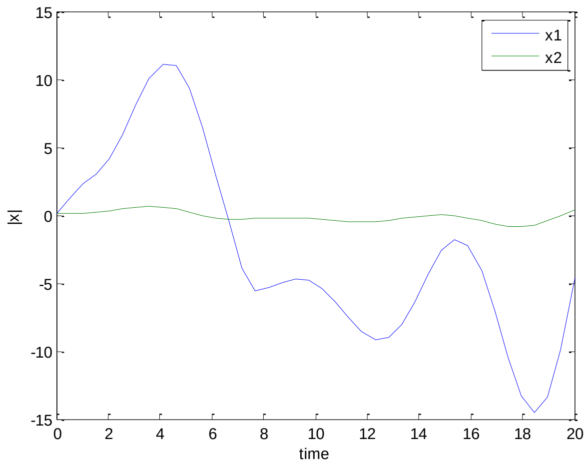

Fig. 3. The chaotic behavior of the NN(Example2).

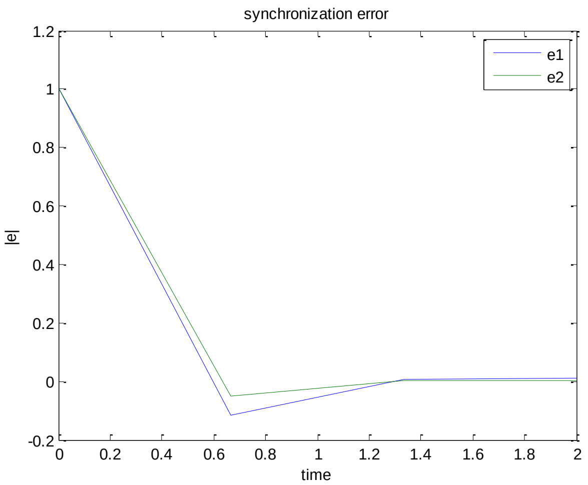

Fig. 4. The synchronization error (Example 2).

V.CONCLUSION

This paper has presented a new sufficient condition to solve the exponential synchronization of a class of delayed chaotic neural networks. The proposed sufficient condition is derived primarily by the Halanay inequality lemma rather than through the use of the Lyapunov functional. The result has indicated that the real symmetric and positive definite controller gain matrix Ω is designed to achieve synchronization.

REFERENCES

[1] Vel Tech Dr. RR & Dr. SR, Hybrid synchronization of liu and lü chaotic systems via adaptive control, International Journal of Advanced Information Technology (IJAIT) Vol. 1, No. 6, December 2011.

[2] Pecora LM, Carroll TL, Synchronization in chaotic systems, Phys Rev Lett 1990;64:821–4.

[3] Pecora LM, Carroll TL, Driving systems with chaotic signals, Phys Rev A 1991;44:2374–83.

[4] Brown R, Kocarev L, A unifying definition of synchronization for dynamical systems, Chaos 2000;10:344–9.

[5] Wang C, Ge SS, Adaptive synchronization of uncertain chaotic systems via backstepping design, Chaos, Solitons and Fractals 2001;12:1199–206.

[6] Hu G, Xiao J, Zheng Z, Chaos control, China: Shanghai Scientific and Technological Education Publishing House; 2000.

[7] Chua LO, Itah M, Kosarev L, Eckert K, Chaos synchronization in Chua's circuits, J Cricuits Syst Comput 1993;3:93–108.

[8] Agiza HN, Yassen MT, Synchronization of Rossler and Chen chaotic dynamical systems using active control., Phys Lett A 2001;278:191–7.

[9] Carpenter MC, Grossberg S, Computing with neural network, Science 1987;37:51–115.

[10] Hopfield JJ, Neural networks and physical systems with emergent collective computational abilities, Proc Natl Acad Sci USA 1982;79:2554–8.

[11] Li JH, Michel AN, Porod W, Qualitative analysis and synthesis of a class of neural networks, IEEE Trans Circ Syst 1988;35:976–86.

[12] Joy M, On the global convergence of a class of functional differential equations with applications in neural network theory, JMath Anal Appl 1999;232:61–81.

[13] Gopalsamy K, Stability and oscillations in delay differential equations of population dynamics, The Netherlands: Kluwer Academic Publishers; 1992

[14] Cao J, Global stability conditions for delay CNNs, IEEE Trans Circ Syst 2001;48:1330–3.

[15] Wang L, Zou X, Exponentially stability of Cohen–Grossberg neural networks, Neural Networks 2002;15:415–22.

[16] Zou F, Nossek JA, Bifurcation and chaos in cellular neural networks, IEEE Trans Circ Syst I 1993;40(3):166–73.

[17] Gilli M, Strange attractors in delayed cellular neural networks, IEEE Trans Circ Syst 1993;40(11):849–53.

[18] Lu HT, Chaotic attractors in delayed neural networks, Phys Lett A 2002;298:109–16.

[19] Chen G, Zhou J, Liu Z, Global synchronization of coupled delayed neural networks with application to chaotic CNN models, Int J Bifurcat Chaos 2004;14:2229–40.

[20] Chao-Jung Cheng, Teh-Lu Liao, Chi-Chuan Hwang, Exponential synchronization of a class of chaotic neural networks, Chaos, Solitons and Fractals 24 (2005) 197–206.

[22] Farouk Chérif, ,Dynamics and Oscillations of GHNNs with Time-Varying Delay, LNCS 7552, pp. 17.24, 2012. Springer-Verlag Berlin Heidelberg 2012.

[23] Farouk Chérif, Existence and global exponential stability of pseudo almost pe-riodic solution for SICNNs with mixed delays,J Appl Math Comput (2012) 39:235.251.